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# The statistics of transport parameters in the system of weakly coupled disordered chains

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**Abstract.** We numerically study the statistics of the transport parameters of the system of disordered chains, mutually coupled by hopping term $t \ll 1$. We find that the system can be described in terms of the random matrix theory with $t$-dependent 'symmetry parameter' $\beta$. In the limit $t \to 0 \beta(t)$ behaves as $\sim t^{\alpha}$ with $\alpha \approx 0.11$.

## 1. Introduction

The application of the random-matrix theory (RMT) [1] enables us to understand many peculiarities and the most important features of the electronic transport in weakly disordered mesoscopic systems [2] (for a review see [3]).

Transport properties of disordered system can be expressed in terms of quantities $z$, defined from the eigenvalues $\wedge_i$ of the matrix $T^{\dagger}T$ ($T$ is the transfer matrix) as $\wedge_i = \exp z_i$. In terms of $z$, the conductance $g$ of the $N$-channel system is given as $g = \sum_i^N \cosh^{-2}(z_i/2)$ [4].

RMT suggests that the parameters $z$ are distributed with probability distribution

$$P_{\beta}(z_1, z_2, \ldots, z_N) = \exp -\left[ \sum_i V(z_i) - \beta \sum_{i<j} u(z_i, z_j) \right] \tag{1}$$

[2]. In (1), $\beta$ is the symmetry parameter ($\beta = 1, 2$ and 4 for the orthogonal, unitary and symplectic ensemble, respectively), $V(z_i)$ is the one-particle 'potential' and $u(x, y)$ is the 'interaction potential'.

Distribution (1) was originally proposed for studies of weakly disordered systems; in this limit, it can be derived from the 'maximum entropy ansatz' [2] or the Dorokhov–Mello–Pereyra–Kumar equation [5, 6]. This enabled us to estimate the form of the potentials $V$ and $u$ [7]. For weak disorder, it is also successful at describing the insulating regime which appears in the very long weakly disordered systems [2, 8, 3].

As the distribution $P(z)$ only depends on a small number of parameters, such as the symmetry parameter $\beta$, mean-free path $l$, the number of channels $N$ and the system size $L$, it explains successfully the universal transport properties, observed in mesoscopic systems, as universal conductance fluctuations [7].

The success of distribution (1) at explaining transport in the weak disorder limit inspired the search of its possible generalization to the more complicated systems [9], and even to

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the description of the metal–insulator transition (MIT) [10]. Of course, in this case some other system parameters enter distribution $P(z)$; thus, in our previous work [11] we have found that the growth of the disorder to its critical value causes a change of the potential $V(z)$ from the quadratic form, $V(z) \sim z^2$, which is typical for the metallic limit, to the dimension-dependent behaviour $V(z) \sim z^d$ in the $d$-dimensional system.

In this paper we discuss another possible generalization of the RMT distribution (1). Here we present our results of the numerical simulation of distribution $P(z)$ of highly anisotropic systems and interpret them in the language of RMT.

The transport in anisotropic systems has recently been studied intensively, for example in the systems of conjugated polymers [12, 13] or in the high-temperature superconductors [15–17].

We consider a system of $N$ disordered wires coupled together by weak hopping term. Typical Hamiltonian of our interest has a form

$$H_A = \sum_{i=1}^{N} H_i + t \sum_{ii'} H_{ii'} \tag{2}$$

where

$$H_i = W \sum_{j=1}^{L} \varepsilon_{ij} |ij\rangle\langle ij| + \sum_{j=1}^{L} |ij\rangle\langle ij+1| + \text{c.c.}$$

is the Hamiltonian of the $i$th disordered chain of the length $L$ ($W$ measures the strength of the disorder and $\varepsilon$'s are random energies), and

$$H_{ii'} = \sum_{j=1}^{L} |ij\rangle\langle i'j|$$

is the hopping term between chains $i$ and $i'$; the choice of pairs $ii'$ defines the topology of the model. Parameter $t$ in (2) defines the strength of the hopping (anisotropy).

Anisotropic systems in the limit of $t \ll 1$ have been studied previously with the aim to describe the MIT [14]. As the critical disorder $W_c \propto \sqrt{t}$ [16] in the limit $t \to 0$, such systems enable the analysis of the MIT in the weak disorder limit. In contrast to the previous works, in this paper we concentrate to the limit of small $t$ while keeping the strength of the disorder $W$ constant. This problem cannot be treated by any weak disorder method even for small $W$. Indeed, the small parameter $W$, used in the weak-disorder expansions (see, e.g. [18]) should now be replaced by $W/t$ which is $\gg 1$ as $t \to 0$.

We study two different regimes: in section 2 we examine how the anisotropy influences the spectrum of the quasi-one-dimensional (Q1D) systems. We argue that the spectrum can be described in the framework of RMT, if the parameter $\beta$ in (1) is allowed to have non-integer, $t$-dependent values. The singular behaviour of $\beta(t)$ in the limit $t \to 0$ is obtained numerically. In section 3 we study the most simple system of two coupled chains in the metallic regime. For a given hopping parameter $t$ we have found the corresponding $\beta = \beta(t)$ for which the numerically calculated distribution of $z$'s coincides with (1). Concluding remarks are given in section 4.

## 2. Quasi-one-dimensional systems

We start our consideration with the Q1D Anderson model, defined on the lattice $M \times M \times L$, $L \gg M$ by Hamiltonian

$$H_\alpha = W \sum_{xyz} \varepsilon_{xyz} |xyz\rangle\langle xyz| + \sum_{[nn]} |xyz\rangle\langle x'y'z'|.$$

$L$ is assumed to be long enough to assure the self-averaging of $z$'s. At the critical point $W = W_c$ of MIT, the spectrum of the rescaled variables $\zeta_i = \frac{M}{L} z_i$, has a form

$$\zeta_i = \zeta_1 \sqrt{1 + \frac{\zeta_1}{2}(i-1)} \tag{3}$$

for $i \leqslant \mathcal{N} \propto M$ and is system-size independent for a given value of $L/M \gg 1$. The spectrum (3) has been found numerically [19, 20] and derived from the distribution (1) of $z$'s under assumptions that the interaction term $u(z_i, z_j) = \log|\cosh z_i - \cosh z_j|$ is the same as in the weak disorder limit, and that the one-particle potential $V(z)$ behaves as $\propto z^3$ [11]. Numerical data [19] confirmed that $\zeta_i \sim \mathcal{O}(1)$, and that the spectrum of $z$'s can be approximate by product of independent Gaussians

$$P(z) = \text{constant} \times \prod_i \exp -W(z_i)$$

with

$$W(z_i) = \left(z_i - \frac{L}{M}\zeta_i\right)^2 / 2\sigma_i^2 \tag{4}$$

and with variance $\sigma_i^2 \propto z_i^{-1}$.

We now introduce the anisotropy by considering four systems of the size $M \times M \times L$, connected to each other by hopping term $\propto t$ which enables electrons to move from the surface of one system to the neighbouring one. The spectrum of $z$'s depends strongly on the value of parameter $t$. Evidently, for $t = 0$, it is fourfold degenerated. Coupling $t > 0$ removes this degeneracy (figure 1). When $t = 1$, the spectrum of the resulting system must be (after rescaling $L \to 2L$) identical with (3). Numerical data show that already for $t \approx 0.1$ the resulting spectrum is almost identical to (3).

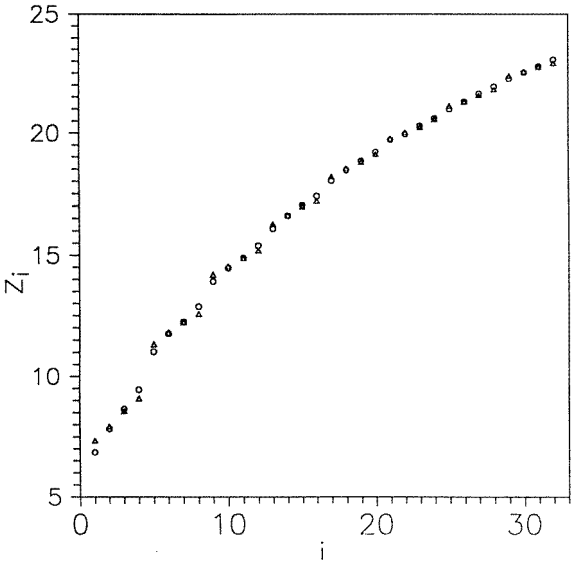


**Figure 1.** The spectrum of system of four bars of the cross section $6 \times 6$, coupled together by small $t = 0.01$ (△) and $t = 0.001$ (○). The unperturbed spectrum ($t = 0$) is four-times degenerated; weak $t$ removes this degeneracy.

The description of the spectrum in the limit of small $t$ is possible using the Coulomb gas analogy [2], in which $z_i$ represents positions of charged particles which move in the potential $V$ and interact via logarithmic interaction $u$. The mean values of $z_i$, $\frac{L}{M}\zeta_i$ represent their stationary positions. Hopping term $t$ causes an additional interaction between charged particles from neighbouring subsystems. As a whole spectrum behaves as that for $2M \times 2M \times L$ Q1D system for $t \to 1$, it is reasonable to assume that $t$-dependent interactions have the standard form known from the RMT

$$u(z_{i\mu}, z_{jv}) = \beta' \sum_{(i\mu)\neq(jnu)} \log|\cosh z_{i\mu} - \cosh z_{jv}| \tag{5}$$

where $\mu, v$ counts the subsystems. The anisotropy $t$ influences only the value of the interacting constant. Evidently, $\beta' \to 1$ for $t \to 1$; in the limit of $t \ll 1$, however, the interaction term must be weaker. As the difference $\Delta_i = \zeta_{i+1} - \zeta_i$ for the 'unperturbed' system is $\mathcal{O}(1)$, it is reasonable to suppose that in the limit $t \ll 1$ the additional interaction influences only the position of 'particles', which lie close to each other; it allows us to neglect the interaction $u(z_{i\mu}, z_{jv})$ for $i \neq j$ and $\mu \neq v$. Then, the distribution $P(z)$ of all $z$'s splits into the form

$$P(\{z_{i\mu}\}) = \prod_i P_i(z_{i\mu}) \tag{6}$$

where

$$P_i(\{z_{i\mu}\}) = \exp -\left[\sum_\mu W(z_{i\mu}) - \beta' \sum_{\mu<v} u(z_{i\mu}, z_{jv})\right]. \tag{7}$$

Potential $W(z_i)$ is given by (4) and the interaction potential is of the form (5) with $\beta' = \beta'(t)$.

The distribution $P_i(z_{i\mu})$ is (6) can be interpreted as that of the system of $N = 4$ chains coupled together by hopping term $t$. As we have neglected the interaction term $u(z_{i\mu}, z_{jv})$ for $i \neq j$ and $\mu \neq v$, these systems are statistically independent from each other.

Owing to (7), the distribution for each such four-chain system still has the form of (1) but with $t$-dependent symmetry parameter $\beta'(t)$. Then, the mutual position $z_{i\mu}$ can be found from the standard RMT:

$$z_n = 2\frac{L}{l} \times \frac{1 + \beta' n - \beta'}{2 + \beta' N - \beta'} \tag{8}$$

where $l$ is the mean-free path [3, 2]. When considering $\beta'$ small, we can expand (8) into the powers of $\beta'$, and obtain

$$z_n \approx z_0 \left(1 + \frac{\beta'}{2}(2n - N - 1\right) \tag{9}$$

which explains the shift of each of four originally degenerate $\zeta$'s in figure 1.

To obtain more quantitative data for the small-$t$ behaviour of $\beta$, we study the system of only $N = 2$ coupled disordered chains. Their Hamiltonian is given by (2) with $N = 2$. Application of (9) to this simple system gives

$$\frac{z_1}{z_0} - 1 = 1 - \frac{z_2}{z_0} \approx \frac{\beta}{2} \qquad \ll 1. \tag{10}$$

In figure 2 we present data for the differences (10) for two chain problem. Data show that the degeneracy $z_1 = z_2 = z_0$ of $z$'s for $t = 0$ is quickly removed when $t > 0$ and the differences $z_2 - z_0$ and $z_0 - z_1$ are approximately equal to each other, which agrees with (10). Thus, the spectrum of our system follows formula (8) with $t$-dependent parameter $\beta$.

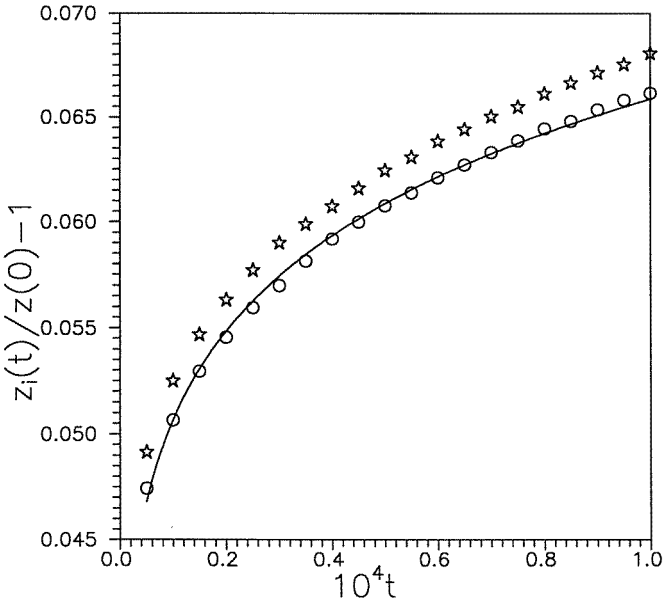


**Figure 2.** $t$-dependence of $|z_1(t)/z_0 - 1|$ and $|z_2(t)/z_0 - 1|$ for very long two-chain system: $L = 50\,000$, $W = 2$, averaging over ensemble of 1000 systems has been performed. The full curve is the power fit $\sim t^{0.11}$.

By comparing relations (9) with numerical data for the difference $\Delta = z_2 - z_1$ we observe the singular behaviour of parameter $\beta$ in the limit $t \to 0$:

$$\beta(t) \propto t^{\alpha} \qquad \alpha \approx 0.11. \tag{11}$$

We have found approximately the same value for $\alpha$ for a two-chain systems with different disorder and energy (figure 3(*a*)) and also for systems with $N = 3$ chains.

To check the validity of relation (11) for more complex systems we also studied the system of $N = 36$ channels $6 \times 6 \times \infty$ with hopping $= 1$ in the $x$ direction and hopping $t$ in the $y$ direction. In the limit $t \ll 1$ this system corresponds to weakly coupled disordered planes. For the differences $z_{i+1} - z_i$ we found again that

$$z_{i+1} - z_i \propto t^{\alpha}$$

with exponent $\alpha$ close to the value refereed in (11) (figure 3(*b*)).

## 3. Metallic phase

In this section we consider a system of weakly disordered short chains (short enough to assure that $z \leqslant 1$ in the absence of mutual coupling) and study how the weak coupling $t$ influences the form of the distribution $P(z)$. On the basis of results of section 2 we conjecture that the last still has form of (1) but with $t$-dependent parameter $\beta$.

In contrast to Q1D systems there is no degeneracy in $z$'s already for $t = 0$. Instead, the spectrum of $z$'s possesses some rigidity typical for the spectrum of random matrices. However, it is a purely statistical effect caused by the ordering of $N$ one-dimensional $z$ for each member of the ensemble; the smallest one, $z_1^{(N)}$, is then distributed according to the

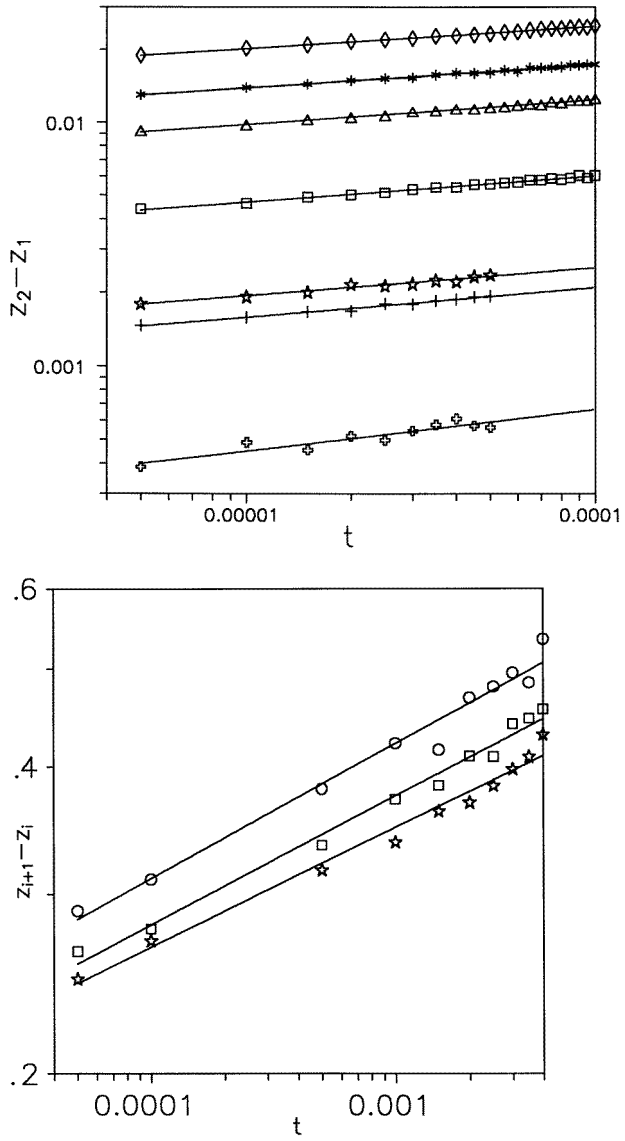


**Figure 3.** $t$-dependence of $\Delta z$ for ($a$) two chains, $L = 5 \times 10^4$, and different disorder $E = 5, 4, 3, 2, 1$ and for energy $E = 1$, $W = 1$, ($b$) Q1D system with cross sections $6 \times 6$ with anisotropy in one direction of the bar. The full curves are power fits $\Delta \propto t^{\alpha}$ with $\alpha \approx 0.11$.

distribution

$$P_1(z_1^{(N)}) = N! \int_0^{\infty} \int_{z_1}^{\infty} \dots \int_{z_{N-1}}^{\infty} \prod_{i=1}^{N} \mathrm{d}_{z_i} \, P(z_i) \delta(z_1^{(N)} - z_1) \tag{12}$$

and similar relations can also be written for higher $z$'s.

$P(z)$ is the distribution of $z$ in the single channel. For weak disorder, it is almost identical to Wigner surmises. We can therefore consider $P(z) \propto z \exp -z^2$, after simple

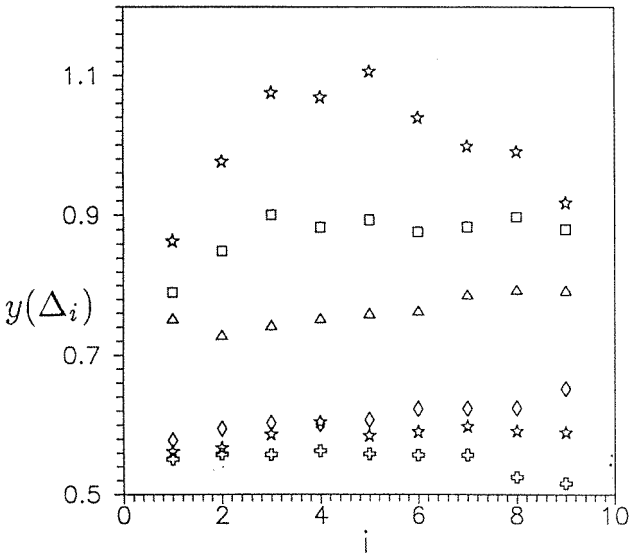


**Figure 4.** $t$-dependence of $y(\Delta)$ for system $10 \times 10$ and for different values of $t$ (from above: $t = 0, 0.1, 0.2, 0.5, 0.7$ and $1.0$).

integration we obtain

$$P_1(x) \propto x \exp -\frac{\pi}{4} N x^2. \tag{13}$$

Thus, the mean value of $z_1$ is smaller by a factor of $N^{-1/2}$ in comparison with that for the one-dimensional chain.

Despite the regularity of the spectra, the statistics of $z$'s differs considerably from that predicted by RMT. Indeed, the distribution of differences $\Delta_i = z_{i+1} - z_i$ should be close to the Poissonian distribution; that means that the ratio

$$y(\Delta_i) = \sqrt{\mathrm{var}\Delta_i / \langle \Delta_i \rangle}$$

is close to 1.

Weak coupling $t$ between chains enables particles to hop from one chain to the next neighbour one and causes the quick decrease of $y(\Delta)$ to the value $\approx 0.522$ which is typical for Wigner surmises. We illustrate this behaviour in figure 4 which presents the $t$-dependence of $y(\Delta_i)$ for the $10 \times 10$ weakly disordered system.

More quantitative studies have been performed for the most simple system consisting of only $N = 2$ chains of the length $L = 40$. The strength of the disorder was chosen such that the localization length exceeds $L$. The distributions of $z_1$ and $\Delta_1$ has been calculated numerically. While the distribution $P(z_1)$ exhibits no qualitative changes with $t$ (as we mentioned above, it is of the form of Wigner surmises already for $t = 0$), distribution $P(\Delta)$ changes from a Poissonian to a Wigner distribution as $t$ grows. To describe this crossover we suppose that the distribution $P(z_1, z_2)$ has the form

$$P_\beta(z_1, z_2) = |z_2 - z_1|^\beta z_1 z_2 \exp -(z_1^2 + z_2^2). \tag{14}$$

Distribution (14) can be obtained from the 'classical' RMT distribution (1) in the limit of small $z$'s and by appropriate rescaling of $z$'s which enable us to avoid parameters such as the mean value of $z$. We suppose that the anisotropy influences only the value of parameter

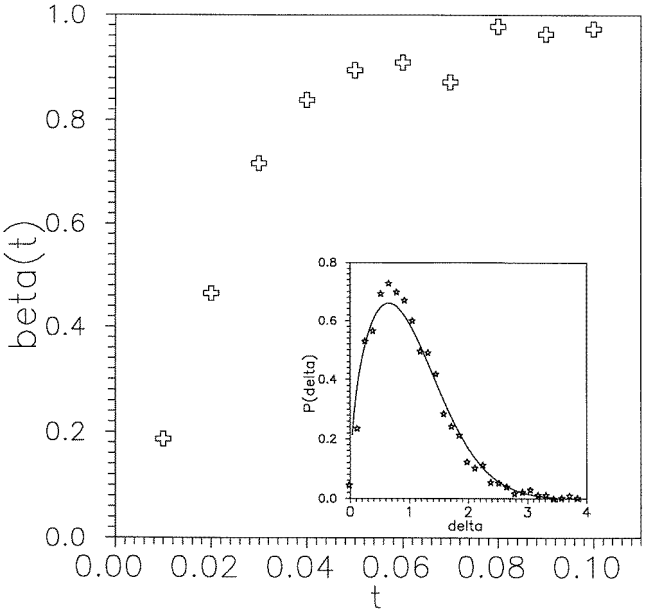


**Figure 5.** $t$-dependence of $\beta$ for $N = 2$ coupled chains. Inset: distribution $P(\Delta)$ for $t = 0.02$ found from numerical simulations, and its comparison with distribution of $\Delta$ which follows from (14) for $\beta = 0.4546$.

$\beta$. For $t = 0$ the system splits into two independent chains, so that $\beta(t = 0) = 0$; and for $t = 1$ we suppose $\beta = 1$.

We use distribution (14) to calculate by numerical integration the distribution $P(\Delta)$ of the difference $\Delta = z_2 - z_1$ for different values of $\beta$. Then, the comparison of the ratio $y_\beta(\Delta)$ with that found from numerical simulation of the two-chain system enables us to find the $t$-dependence of $\beta$ in distribution (14). The obtained function $\beta(t)$ is shown in figure 5. As expected, $\beta$ grows quickly as $t$ increases; in fact, already for $t \approx 0.1$ it is close to its limiting value. This is consistent with results discussed in section 3.

The inset of figure 5 presents the comparison of the distribution of $\Delta$ for $t = 0.02$ with that calculated from (14) for the corresponding value of $\beta = 0.4546$.

## 4. Conclusion

In conclusion, we have studied the role of the anisotropy on the statistical properties of the system of coupled chains and shown that the statistics of such systems could be described by a probability distribution similar to that from RMT but with non-integer symmetry parameters $\beta$. We study the dependence of $\beta$ on the anisotropy parameter $t$. In the limit $t \to 0$, we found, from the studies of weakly coupled long chains, that $\beta \propto t^\alpha$ with $\alpha \approx 0.11$. The same exponent $\alpha$ also characterizes the $t$-dependence of differences $z_{i+1} - z_i$ for system of weakly coupled disordered plains.

In the metallic regime, the $t$-dependence of $\beta$ has been found from the studies of the whole distribution of the difference $z_2 - z_1$ of two-chain system. For larger systems we have shown that the distributions $P(z_{i+1} - z_i)$ possess the same transition from the Poissonian to the Wigner statistics as that for the two-chain problem. It is therefore reasonable to suppose

that the $\beta(t)$ relation has qualitatively similar behaviour also for general $N$-chain systems.

Present results suggest the possibility of generalizing the random-matrix treatment also to studies of anisotropic systems. They enable us to also consider another transition from the metallic into the non-metallic regime; different to previous treatment [10, 11] where the localized regime has been achieved by appropriate change of the one-particle potential, the same is achieved just by decrease of parameter $\beta$.

Our consideration provides us also with the possible physical interpretation of the random-matrix statistics with non-integer 'symmetry parameter' $\beta$.

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## References

[1] Mehta M L 1991 *Random Matrices* (New York: Academic)
[2] Pichard J-L 1991 *Quantum Coherence in Mesoscopic Systems (NATO ASI Series B 254)* ed B Kramer (New York: Plenum) p 369
[3] Beenakker C W J 1997 *Rev. Mod. Phys.* **69** 731
[4] Pichard J-L 1984 *PhD Thesis* 2858 University of Paris, Orsay
[5] Dorokhov O N 1983 *Solid State Commun.* **46** 605
Dorokhov O N 1984 *Solid State Commun.* **51** 381
[6] Mello P A, Pereyra P and Kumar N 1988 *Ann. Phys., NY* **181** 290
[7] Beenakker C W J and Rejaei B 1994 *Phys. Rev.* B **49** 7499
[8] Muttalib K A 1990 *Phys. Rev. Lett.* **65** 745
[9] Mello P A and Tomsovic' S 1992 *Phys. Rev.* B **46** 15 963
[10] Chen Y, Ismail M E H and Muttalib K A 1992 *J. Phys.: Condens. Matter* **4** L417
Muttalib K A, Chen Y, Ismail M E H and Nicopoulos V N P 1993 *Phys. Rev. Lett.* **71** 471
[11] Markoš P 1995 *J. Phys.: Condens. Matter* **7** 8316
[12] Prigodin V N and Efetov K B 1993 *Phys. Rev. Lett.* **70** 2932
[13] Markoš P and Evangelou S N 1996 *Ann. Phys.* **5** 526
[14] Chalker J T and Bernhardt M 1993 *Phys. Rev. Lett.* **70** 982
[15] Panagiotides N A, Evangelou S N and Theodorou G 1994 *Phys. Rev.* B **49** 14 122
[16] Zambetaki I, Qiming Li, Economou E N and Soukoulis C M 1996 *Phys. Rev. Lett.* **76** 3614
[17] Evangelou S N *et al* 1997 unpublished
[18] Derrida B, Mecheri K and Pichard J-L 1987 *J. Physique* **48** 733
[19] Markoš P and Kramer B 1993 *Phil. Mag.* B **68** 357
[20] Markoš P 1995 *NATO ASI Series* B **291** 99